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M.Sc. (Final) DEGREE EXAMINATION, DECEMBER 2008.

Second Year

Mathematics

Paper I — TOPOLOGY AND FUNCTIONAL ANALYSIS

Time : Three hours **Maximum : 100 marks**

Answer any FIVE questions choosing atleast Two from each of Part A and Part B.

PART A

1. (a) Let X be a topological space. Then prove that (i) any intersection of closed sets in X is closed and (ii) any finite union of closed sets in X is closed.

(b) Prove that every separable metric space is second countable.

2. (a) Prove that any closed subspace of a compact space is compact.

(b) State and prove the Heine-Borel theorem.

3. (a) Prove that a metric space is sequentially compact if and only if it has Bolzano-Weierstrass Property.

(b) State and prove Lebesgue's covering lemma.

4. (a) Prove that a one-to-one continuous mapping of a compact space onto a Hausdorff space is a homomorphism.

(b) State and prove Urysohn's Lemma.

PART B

5. (a) Prove that the product of any non-empty class of connected spaces is connected.

(b) Let X be a topological space. If $\{A_i\}$ is a non-empty class of connected subspaces of X such that $\bigcap_i A_i$ is nonempty, then prove that $A = \bigcup_i A_i$ is also a connected subclass of X.

6. (a) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in the quotient space N/M is defined by

$$\|x + M\| = \inf \{\|x + m\| : m \in M\}$$

then prove that N/M is a normed linear space. Further prove that if N is a Banach space then so is N / M.

(b) If N is normed linear space and x_0 is a non-empty vector in N, then prove that there exists a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.

7. State and prove open mapping theorem.

8. (a) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

(b) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$.

9. (a) If $\{e_i\}$ is an orthonormal set in a Hilbert space H, and if x is an arbitrary vector in H, then prove that $x - \Sigma(x, e_i)e_i \perp e_i$ for each j.

(b) Let H be a Hilbert space, and let f be an arbitrary functional in H^*. Then prove that there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

10. (a) If T is an operator on H, then prove that T is normal $\Leftrightarrow$ its real and imaginary parts commute.

(b) If P and Q are the projections on closed linear subspaces M and N of H, then prove that $M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$.

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M.Sc. (Final) DEGREE EXAMINATION, DECEMBER 2008.

Second Year

Mathematics

Paper II — MEASURE AND INTEGRATION

Time : Three hours **Maximum : 100 marks**

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Let A be an algebra of subsets and $\{A_i\}$ be a sequence of sets in A. Then prove that there is a sequence $\{B_i\}$ of sets in A such that $B_n \cap B_m = \phi$ for $n \neq m$ and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.

(b) State and prove Heine–Borel theorem.

2. (a) Prove that the outer measure of an interval is its length.

(b) If E_1 and E_2 are measurable, then prove that $E_1 \cup E_2$ is also measurable.

3. (a) Let $\{E_i\}$ be a sequence of measurable sets. Then prove that $m(\bigcup E_i) \leq \Sigma m E_i$. If the sets E_n are pairwise disjoint, then $m(\bigcup E_i) = \Sigma\, m E_i$.

(b) State and prove Littlewood's third principle.

4. (a) Let f be defined and bounded on a measurable set E with mE finite. In order that

$\inf_{f \leq \psi} \int_E \psi(x)\, dx = \sup_{f \geq \phi} \int_E \phi(x)\, dx$ for all simple functions ϕ and ψ, prove that it is necessary and sufficient that f be measurable.

(b) If f and g are bounded measurable functions defined on a set E of finite measure. Then prove that

(i) $\int_E (af + bg) = a\int_E f + b\int_E g$.

(ii) If $f = g$ a.e., then $\int_E f = \int_E g$.

5. (a) State and prove Fatou's lemma.

(b) Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Then prove that there is a subsequence $\{f_m\}$ that converges to f almost everywhere.

6. (a) State and prove Vitali lemma.

(b) If f is integrable on $[a,b]$ and $\int_a^x f(t)\,dt = 0$ for all $x \in [a, b]$, then prove that $f(t) = a$, a.e. in $[a,b]$.

7. (a) State and prove Holder inequality.

(b) Let g be an integrable function on $[0, 1]$, and suppose that there is a constant M such that $\left|\int f g\right| \le M \| f \|_g$ for all bounded measurable functions. Then prove that g is in L^p and $\|g\|_q \le M$.

8. (a) If $E_i \in B$, $\mu E_1 < \infty$ and $E_i \supset E_{i+1}$, then prove that $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu E_n$.

(b) State and prove Hahn decomposition theorem.

9. State and prove Radon-Nikodym theorem.

10. (a) Prove that the set function μ^* is an outer measure.

(b) The class **B** of μ^* - measurable sets is a σ – algebra. If $\bar{\mu}$ is μ^* restricted to **B**, then prove that $\bar{\mu}$ is a complete measure on **B**.

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M.Sc. (Final) DEGREE EXAMINATION, DECEMBER 2008.

Second Year

Mathematics

Paper III — ANALYTICAL NUMBER THEORY AND GRAPH THEORY

Time : Three hours **Maximum : 100 marks**

Answer any FIVE out of the given TEN questions selecting atleast TWO from each Part.

PART A

1. If $x \geq 1$, prove that the following asymptotic formulas :

(a) $\sum_{n \leq x} \frac{1}{n} = \log x + c + O\left(\frac{1}{x}\right)$.

(b) $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$ if $s > 0, s \neq 1$.

(c) $\sum_{n > x} \frac{1}{n^s} = O(x^{1-s})$ if $s > 1$

(d) $\sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$ if $\alpha \geq 0$.

2. (a) For all $x \geq 1$, prove that

$$\sum_{n \leq x} T(n) = \frac{1}{2}\zeta(2)\, x^2 + O(x \log x).$$

(b) For $x \geq 2$, prove that

$$\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x).$$

where the sum is extended overall primes $\leq x$.

3. (a) State and prove Abel's identity.

(b) For every integer $n \geq 2$ prove that

$$\frac{1}{6}\frac{n}{\log n} < \pi(n) < 6\frac{n}{\log n}.$$

4. (a) Show that there is a constant A such that

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + A + O\left(\frac{1}{\log x}\right) \text{ for all } x \geq 2.$$

(b) Show that

$$\lim_{x \to \infty}\left(\frac{M(x)}{x} - \frac{H(x)}{x \log x}\right) = 0.$$

PART B

5. (a) Show that a graph G is disconnected if and only if its vertex set V can be partitioned into two non empty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

(b) Give a brief account of Konigsberg Bridge problem.

6. (a) Prove that any two simple connected graphs with n vertices, all of degree two, are isomorphic.

(b) In a complete graph with n vertices, show that there are $(n-1)/2$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3.

7. (a) Show that a graph G with n vertices, $n-1$ edges, and no circuits is connected.

(b) Show that the number of labeled trees with n vertices $(n \geq 2)$ is n^{n-2}.

8. (a) Show that every cut-set in a connected graph G must contain atleast one branch of every spanning tree of G.

(b) Show that the ring sum of any two cut-sets in a graph is either a third cut-set or an edge - disjoint union of cut-sets.

9. (a) Show that a connected planar graph with n vertices and e edges has $e-n+2$ regions.

(b) Show that a graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

10. (a) Show that the ring sum of two circuits in a graph G is either a circuit or an edge-disjoint union of circuits.

(b) Explain the concept of vector space associated with graph in detail.

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M.Sc. (Final) DEGREE EXAMINATION, DECEMBER 2008.

Second Year

Mathematics

Paper VI — RINGS AND MODULES

Time : Three hours **Maximum : 100 marks**

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Let $(S, \wedge)$ be a semigroup satisfying commutative and Idempotent laws. For a, b in S. Define $a \leq b$ if and only if $a \wedge b = a$. Then $(S, \leq)$ is a semilattice.

(b) Show that in any Boolean ring we have the identities (i) $a + a = 0$ and (ii) $a \cdot b = b \cdot a$.

2. (a) If 'f' is a homomorphism from R to S and 'g' is a homomorphism from S to T. Then we have the following :

(i) 'f' and 'g' are monomorphism, implies '$g \circ f$' is monomorphism.

(ii) 'f' and 'g' are epimorphisms implies '$g \circ f$' is epimorphisms.

(iii) $g \circ f$ is epimorphism implies g is epimorphism.

(b) There is a one-to-one correspondence (that is a bijection) between the ideals K and the congruence relation θ of a ring R such that $r - r' \in E \Leftrightarrow r \,\theta\, r'$.

3. (a) Prove that the following statements are equivalent :

(i) R is isomorphic to a finite direct product if rings R_i (for $l = 1, 2, \cdots n$).

(ii) There exist central orthogonal idempotent $e_i \in R$ such that $1 = \sum_{i \in 1} e_i$ and $e_i R \cong R_i$.

(iii) R is a finite direct sum of ideas $K_i \cong R_i$.

(b) If B and C are sub module of A then $(B+C)/B$ isomorphic to $C/(B \cap C)$.

4. (a) (i) Prove that the proper ideal M of the commutative ring R is maximal $\Leftrightarrow$ for all $r \notin M$ there exist $x \in R$ such that $1 - r\,x \in M$.

(ii) Prove that the proper ideal M of the commutative ring R is prime $\Leftrightarrow$ for all element a and b, $ab \in M \Rightarrow a \in M$ or $b \in M$.

(b) Prove that A commutative ring R is semi prime $\Leftrightarrow$ it is isomorphic to a sub ring of direct product of integral domains.

5. (a) (i) Prove that if R is sub directly irreducible and semi prime then R is a field.

(ii) Prove that θ is an congruence relation on the system $(F, 0, 1, -, +, .)$.

(b) Prove that if R is a commutative ring then $Q(R)$ is rationally complete.

6. (a) State and prove Jacobson-Density theorem.

(b) Prove that the following statements concerning the ring R are equivalent.

(i) Every right R–module is completely reducible.

(ii) R_R is completely reducible.

(iii) Every left R–module is completely reducible.

(iv) R^R is completely reducible.

7. State and Wederburn-Artin theorem.

8. (a) State and prove Hilbert Basis theorem.

(b) Prove that if R is a regular ring then every finite by generated right ideal is principal.

9. (a) (i) Prove that Every module is isomorphic to a factor of a free module.

(ii) If M is projective $\Leftrightarrow$ every epimorphism $\pi : B \to M$ is direct.

(b) Let $M = \sum_{i \in I} M_i$ be a direct sum of sub modules M_i. Assume that not only M_i but all sub modules of each M_i are projective. Show that any sub modules N of M is isomorphic to a direct sum $\sum_{i \in I} N_i$ where $N_i \subset M_i$ for every $i \in I$.

10. (a) State and prove Baer's criteria for injectivity.

(b) Prove that every module is isomorphic to a sub module of an injective module.
